Faster than Lyapunov decays of classical Loschmidt echo

Gregor Veble[1,2] and Tomaž Prosen[1]

[1] *Physics Department, FMF, University of Ljubljana, Ljubljana, Slovenia*
[2] *Center for Applied Mathematics and Theoretical Physics, University of Maribor, Slovenia*
veble@fiz.uni-lj.si, prosen@fiz.uni-lj.si

Abstract

We show that the composition of perturbed forward and unperturbed backward hamiltonian evolution can be treated as a time-dependent hamiltonian system. For strongly chaotic systems we derive a cascade of exponential decays for the classical Loschmidt echo, starting with the leading Lyapunov exponent, followed by a sum of two largest exponents, etc.

The *Classical Loschmidt Echo* (CLE) has been defined [1, 2, 3, 4] as

$$F(t) = \int \mathrm{d}^N \boldsymbol{x} \rho(\boldsymbol{x}) \rho_{\mathrm{E}}(\boldsymbol{x}, t) \tag{1}$$

where $\rho(\boldsymbol{x})$ is an L^2 normalized non-negative initial density in $N = 2d$-dimensional classical phase space with coordinates $\boldsymbol{x}$, and $\rho_{\mathrm{E}}(\boldsymbol{x}, t)$ is its image after the *echo-dynamics*, i.e. a composition of a hamiltonian flow with a slightly perturbed time-reversed hamiltonian flow.

For short times it has been found numerically [2] that CLE decays exponentially with the rate given by the Lyapunov exponent. No classical mechanism for this phenomenon has been given, apart from the necessary correspondence with the quantum Loschmidt echo (QLE) for which a semiclassical theory of Lyapunov decay exists [5]. Here we report a surprising analytical result for the case of localized initial densities and for sufficiently weak perturbations. In many-dimensional systems, a cascade of Lyapunov decays is predicted, with the exponents which are given as consecutive sums of largest few Lyapunov exponents.

The propagation of classical densities in phase space is governed by the unitary Liouville evolution $\hat{U}_\delta(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \hat{U}_\delta(t) = \hat{L}_{H_\delta(\boldsymbol{x},t)} \hat{U}_\delta(t) \tag{2}$$

where $\hat{L}_{A(\boldsymbol{x},t)} = (\boldsymbol{\nabla} A(\boldsymbol{x}, t)) \cdot \mathbf{J} \boldsymbol{\nabla}$, A is any observable, and

$$H_\delta(\boldsymbol{x}, t) = H_0(\boldsymbol{x}, t) + \delta V(\boldsymbol{x}, t), \tag{3}$$

is a generally time-dependent family of Hamiltonians with perturbation parameter δ. Matrix $\mathbf{J}$ is the usual symplectic unit. Similarly, $\mathrm{d}\hat{U}_\delta^\dagger(t)/\mathrm{d}t = -\hat{U}_\delta^\dagger(t) \hat{L}_{H_\delta(\boldsymbol{x},t)}$. The *classical echo propagator* that composes perturbed forward evolution with the unperturbed backward evolution is also unitary and is given by

$$\hat{U}_{\mathrm{E}}(t) = \hat{U}_0^\dagger(t)\, \hat{U}_\delta(t). \tag{4}$$

Using eqs. (2,3,4) and writing $\hat{U}_\delta(t) = \hat{U}_0(t) \hat{U}_{\mathrm{E}}(t)$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \hat{U}_{\mathrm{E}}(t) = \left\{ \hat{U}_0^\dagger(t) \hat{L}_{\delta V(\boldsymbol{x},t)} \hat{U}_0(t) \right\} \hat{U}_{\mathrm{E}}(t). \tag{5}$$

The classical phase space evolution is governed by characteristics that are simply the classical phase space trajectories, so the action of the evolution operator on any phase space density is given as $\hat{U}_0(t)\ \rho = \rho \circ \boldsymbol{\phi}_t^{-1}$, where ϕ_t^{-1} denotes the backward (unperturbed) phase space flow from time t to 0. Similarly, $\hat{U}_0^\dagger(t)\ \rho = \rho \circ \boldsymbol{\phi}_t$, where ϕ_t represents the forward phase space flow from time 0 to time t. Here and in the following we assume the dynamics to start at time 0.

We note that echo-dynamics (4) can be treated as Liouvillian dynamics in *interaction picture*, since [6]

$$\left\{\hat{U}_0^\dagger(t)\hat{L}_{A(\boldsymbol{x},t)}\hat{U}_0(t)\rho\right\}(\boldsymbol{x}) = \left\{\hat{L}_{A(\boldsymbol{\phi}_t(\boldsymbol{x}),t)}\ \rho\right\}(\boldsymbol{x}). \tag{6}$$

This extends eq. (5) to form (2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}_{\mathrm{E}}(t) = \hat{L}_{H_{\mathrm{E}}(\boldsymbol{x},t)}\ \hat{U}_{\mathrm{E}}(t) \tag{7}$$

where the *echo Hamiltonian* is given by

$$H_{\mathrm{E}}(\boldsymbol{x},t) = \delta V\left(\boldsymbol{\phi}_t(\boldsymbol{x}),t\right). \tag{8}$$

Trajectories of the echo-flow are given by Hamilton equations

$$\dot{\boldsymbol{x}} = \mathbf{J}\ \boldsymbol{\nabla} H_{\mathrm{E}}(\boldsymbol{x},t). \tag{9}$$

At this point we limit our discussion only to time independent original Hamiltonians and perturbations.

Inserting (8) into eq. (9) yields

$$\dot{\boldsymbol{x}} = \delta\mathbf{J}\boldsymbol{\nabla}_{\boldsymbol{x}} V(\boldsymbol{\phi}_t(\boldsymbol{x})) = \delta\mathbf{J}\mathbf{M}_t^T(\boldsymbol{x})(\boldsymbol{\nabla} V)(\boldsymbol{\phi}_t(\boldsymbol{x})). \tag{10}$$

Here we have introduced the *stability matrix* $\mathbf{M}_t(\boldsymbol{x})$, $[\mathbf{M}_t(\boldsymbol{x})]_{i,j} = \partial_j[\boldsymbol{\phi}_t(\boldsymbol{x})]_i$. From now on we assume that the flow $\boldsymbol{\phi}_t$ is Anosov. To understand the dynamics (10) we need to explore the properties of $\mathbf{M}_t$. We start by writing the matrix $\mathbf{M}_t^T(\boldsymbol{x})\mathbf{M}_t(\boldsymbol{x}) = \sum_j e^{2\lambda_j t} d_j^2(\boldsymbol{x},t)\boldsymbol{v}_j(\boldsymbol{x},t)\otimes\boldsymbol{v}_j(\boldsymbol{x},t)$ expressed in terms of orthonormal eigenvectors $\boldsymbol{v}_j(\boldsymbol{x},t)$ and eigenvalues $d_j^2(\boldsymbol{x},t)\exp(2\lambda_j t)$ (see [6] for details). Similarly, the matrix $\mathbf{M}_t(\boldsymbol{x})\mathbf{M}_t^T(\boldsymbol{x}) = \sum_j e^{2\lambda_j t} c_j^2(\boldsymbol{x}_t,t)\boldsymbol{u}_j(\boldsymbol{x}_t,t)\otimes\boldsymbol{u}_j(\boldsymbol{x}_t,t)$, where $\boldsymbol{x}_t = \boldsymbol{\phi}_t(\boldsymbol{x})$, has the same eigenvalues $[c_j^2(\boldsymbol{x}_t,t) \equiv d_j^2(\boldsymbol{x},t)]$, and its eigenvectors depend on the final point $\boldsymbol{x}_t$ only, as the matrix in question can be related to the backward evolution. The vectors $\{\boldsymbol{u}_j(\boldsymbol{x}_t)\}$, $\{\boldsymbol{v}_j(\boldsymbol{x})\}$, constitute left, right, part, respectively, of the *singular value decomposition* of $\mathbf{M}_t(\boldsymbol{x})$,

$$\mathbf{M}_t(\boldsymbol{x}) = \sum_{j=1}^{N}\exp(\lambda_j t)\ \boldsymbol{e}_j(\boldsymbol{\phi}_t(\boldsymbol{x}))\otimes\boldsymbol{f}_j(\boldsymbol{x}) \tag{11}$$

assuming that the limits $\boldsymbol{e}_j(\boldsymbol{x}) = \lim_{t\to\infty} c_j(\boldsymbol{x},t)\boldsymbol{u}_j(\boldsymbol{x},t)$, $\boldsymbol{f}_j(\boldsymbol{x}) = \lim_{t\to\infty} d_j(\boldsymbol{x},t)\boldsymbol{v}_j(\boldsymbol{x},t)$ exist. Rewriting eq. (10) by means of eq. (11) we obtain

$$\dot{\boldsymbol{x}} = \delta\sum_{j=1}^{N}\exp(\lambda_j t)\ W_j(\boldsymbol{\phi}_t(\boldsymbol{x}))\ \boldsymbol{h}_j(\boldsymbol{x}) \tag{12}$$

where $\boldsymbol{h}_j(\boldsymbol{x}) = \mathbf{J}\boldsymbol{f}_j(\boldsymbol{x})$, and introducing new observables

$$W_j(\boldsymbol{x}) = \boldsymbol{e}_j(\boldsymbol{x})\cdot\boldsymbol{\nabla} V(\boldsymbol{x}). \tag{13}$$

For small perturbations the echo trajectories remain close to initial point $\boldsymbol{x}(0)$ for times large in comparison to the internal dynamics of the system (t_{e}, Lyapunov times, decay of correlations, etc), and in this regime the echo evolution can be linearly decomposed along different independent directions $\boldsymbol{h}_j(\boldsymbol{x}(0))$

$$\boldsymbol{x}(t) = \boldsymbol{x}(0) + \sum_{j=1}^{N} y_j(t)\ \boldsymbol{h}_j(\boldsymbol{x}(0)). \tag{14}$$

Inserting (14) into (12) we obtain for each direction $\boldsymbol{h}_j$

$$\dot{y}_j = \delta \ \exp(\lambda_j t) W_j(\boldsymbol{\phi}_t(\boldsymbol{x})). \tag{15}$$

For *stable* directions with $\lambda_j < 0$, clearly after a certain time the variable y_j becomes a constant of the order δ.

For *unstable* directions with $\lambda_j > 0$, we introduce z_j as $y_j = \delta \exp(\lambda_j t) z_j$ and rewrite the above equation as

$$\dot{z}_j + \lambda_j z_j = W_j(\boldsymbol{\phi}_t(\boldsymbol{x})). \tag{16}$$

The right hand side of this equation is simply the evolution of the observable W_j starting from a point in phase space $\boldsymbol{x} = \boldsymbol{x}(0)$. Due to assumed ergodicity of the flow $\boldsymbol{\phi}_t$, $W_j(\boldsymbol{\phi}_t(\boldsymbol{x}))$ has well defined and *stationary* statistical properties such as averages and correlation functions. Thus the solution $z_j(t)$ of the linear damped equation (16) has also stationary statistics and well defined time- and δ-independent probability distribution $P_j(z_j)$.

Going back to the original coordinate y_j we obtain its distribution as $K_j(y_j) = P_j(z_j)\mathrm{d}z_j/\mathrm{d}y_j$, or $K_j(y_j) = P_j\left(\exp(-\lambda_j t) y_j/\delta\right) \exp(-\lambda_j t)/\delta$. This probability distribution tells us how, on average, points within some initial (small) phase space set of characteristic diameter ν spread along locally well defined unstable Lyapunov direction j and therefore represents an averaged kernel of the evolution of such densities along this direction. Starting from the initial localized density ρ_0, of small width ν such that the decomposition (14) does not change appreciably along ρ_0, the echo dynamics for densities solves as $\rho_t(\boldsymbol{y}) = \int \mathrm{d}^N \boldsymbol{y}' \rho_0(\boldsymbol{y}') \prod_j K_j(y_j - y_j')$. For stable directions j we set $K_j(y_j) = \delta(y_j)$, as the shift of y_j (of order δ) can be neglected as compared to unstable directions. This also implies that the assumption $\delta \ll \nu$ is necessary in order to get any echo at all after not too short times. CLE (1) can now be written as $F(t) = \int \mathrm{d}^N \boldsymbol{y} \ \rho_0(\boldsymbol{y}) \ \rho_t(\boldsymbol{y})$. As long as the width ν_j of ρ_0 along the unstable direction j is much larger than the width of the kernel K_j, there is no appreciable contribution to the fidelity decay in that direction. At time

$$t_j = (1/\lambda_j) \log(\nu_j/(\delta \gamma_j)), \tag{17}$$

where γ_j is a typical width of the distribution P_j, the width of the kernel is of the order of the width of the distribution along the chosen direction. After that time, the overlap between the two distributions along the chosen direction starts to decay with the same rate as the value of the kernel in the neighborhood of $y_j = 0$, which is $\propto \exp(-\lambda_j t)$. The total overlap decays as

$$F(t) \approx \prod_{j;\ t_j < t} \exp\left[-\lambda_j (t - t_j)\right], \tag{18}$$

where only those unstable directions contribute to the decay for which $t_j > t$. As the time t_j is shorter the higher the corresponding Lyapunov exponent λ_j, fidelity will initially decay with the largest Lyapunov exponent λ_1. In chaotic systems with more than two degrees of freedom we, however, expect to observe an increase of decay rate after the time t_2, etc.

Our theory therefore predicts decays which are exponential but faster than Lyapunov. In the case of well separated individual Lyapunov exponents the decay is expected to go through a cascade of increasing decay rates given by (18), whereas in the loxodromic case $\lambda_1 = \lambda_2$ the rate is $2\lambda_1$. We illustrate this numerically for 4D cat maps [7]: $\boldsymbol{x}' = \mathbf{C}\boldsymbol{x} \pmod 1$, $\boldsymbol{x} \in [0,1)^4$, and

$$\mathbf{C}_{\mathrm{d-h}} = \begin{bmatrix} 2 & -2 & -1 & 0 \\ -2 & 3 & 1 & 0 \\ -1 & 2 & 2 & 1 \\ 2 & -2 & 0 & 1 \end{bmatrix}, \ \mathbf{C}_{\mathrm{lox}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & 1 \\ -1 & -1 & -2 & 0 \end{bmatrix}$$

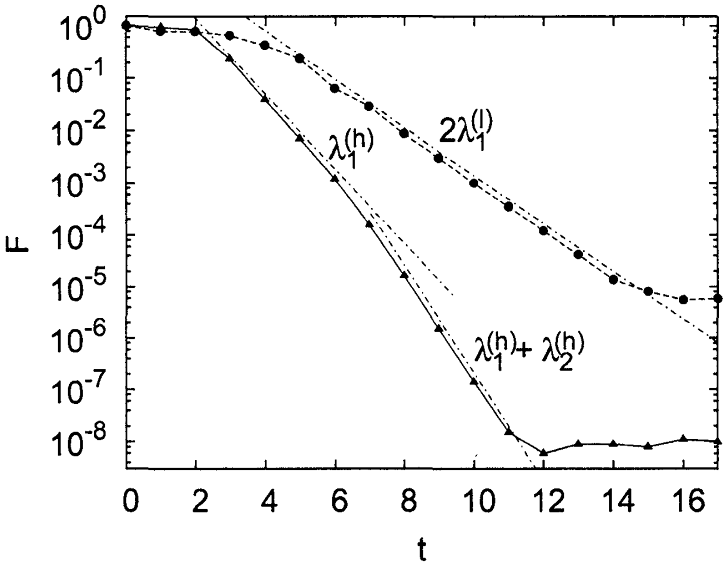

Figure 1: CLE for two examples of 4D cat maps perturbed as explained in text. Triangles refer to doubly-hyperbolic case where initial set was a 4-cube $[0.1, 0.11]^4$, and $\delta = 2 \cdot 10^{-4}$, whereas circles refer to loxodromic case where initial set was $[0.1, 0.15]^4$, and $\delta = 3 \cdot 10^{-3}$. In both cases initial density was sampled by 10^9 points. Chain lines give exponential decays with theoretical rates, $\lambda_1 = 1.65, \lambda_1 + \lambda_2 = 2.40$ (doubly-hyperbolic), and $2\lambda_1 = 1.06$ (loxodromic).

are two examples representing the doubly-hyperbolic and loxodromic case. Matrix $\mathbf{C}_{\mathrm{d-h}}$ has the unstable eigenvalues $\approx 5.22, 2.11$, while the large eigenvalues of $\mathbf{C}_{\mathrm{lox}}$ are $\approx 1.70 \exp(\pm i1.12)$. The perturbation for both cases was done by performing an additional mapping at each timestep $\bar{x}_1 = x'_1 + \delta \sin(2\pi x_3) \pmod 1$, $\bar{x}_{2,3,4} = x'_{2,3,4}$. In figure 1 we show the two types of decay which agree with theoretical predictions.

In conclusion, we have developed a theory for short-time decay of CLE based on classical interaction picture. Our theory predicts several new phenomena, in particular a cascade of exponential decays in systems with more than one unstable direction and doubly-Lyapunov decay for the particular case of loxodromic stability.

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